

MINIMALITY OF A KIND OF PSEUDO-UMBILICAL TOTALLY REAL SUBMANIFOLDS IN NON-FLAT COMPLEX SPACE FORMS

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ABSTRACT. In this paper, by studying the position of umbilical normal vectors in the normal bundle, we prove that pseudo-umbilical totally real submanifolds with flat normal connection in non-flat complex space forms must be minimal.

1. Introduction

Let $\tilde{M}^{n+p}(c)$ be a complex $(n+p)$ -dimensional complex space form endowed with the Fubini-Study metric of constant holomorphic curvature c . An n -dimensional submanifold M^n in $\tilde{M}^{n+p}(c)$ is called totally real if the complex structure J of $\tilde{M}^{n+p}(c)$ carries each tangent space of M^n into its corresponding normal space. Specially, M^n is called Lagrangian if $p = 0$. This kind of submanifolds appear naturally in the context of classical mechanics and mathematical physics and have been studied by many geometers. For instance, Chen [1, 2, 3, 4, 5, 6] have classified Lagrangian surfaces of constant curvature in complex space forms. Shu [7] proved some integral inequalities of Simons' type for n -dimensional compact Extremal Lagrangian submanifolds in complex space forms and gave some rigidity and characterization theorems for general complex co-dimension p . Chen-Ogiue [8] first studied minimal totally real submanifolds. Yano-Kon [9, 10] studied totally real submanifolds satisfying certain conditions on the second fundamental form and provided some basic examples. After that, as a generalization of the minimal ones, totally real submanifolds with parallel mean

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curvature vector field have been studied (see, for example, [11], [12], [13]).

As we all know, pseudo-umbilical submanifolds can be viewed as another natural generalization of the minimal case. From Chen's classification result of slumbilical submanifolds in complex space forms (see [14]), one can construct examples of pseudo-umbilical totally real submanifolds in $\tilde{M}^{n+p}(c)$. However, the class of such submanifolds is too wide to classify, so it is reasonable to study pseudo-umbilical totally real submanifolds under some additional conditions. For example, the author [15] proved that complete pseudo-umbilical Lagrangian surfaces in a complex projective plane must be minimal, and [16] proved that a pseudo-umbilical totally real submanifold M^n with flat normal connection in a complex projective space $\mathbb{C}P^{n+p}$ is minimal if $n = 2$ or $p = 0$. The main purpose of this paper is to show that the latter result also holds for general n and general p , and the ambient space can be assumed to be any non-flat complex space forms. We state it as the following theorem.

Theorem 1.1. *Let M^n be an n -dimensional pseudo-umbilical totally real submanifold in a complex space form $\tilde{M}^{n+p}(c)$ ($c \neq 0$). If the normal connection is flat, then M^n must be minimal.*

Remark 1.1. To prove the above theorem, we characterize the umbilical normal vector by its position in the normal bundle in Section 3 (see Proposition 3.1). This characterization, on the one hand, can provide suitable moving frames for us in the proof of the main theorem. On the other hand, it also implies that there exist no totally umbilical totally real submanifolds whose normal connections are flat in non-flat complex space forms (see Corollary 3.2).

2. Preliminaries

Let M^n be an n -dimensional totally real submanifold in $\tilde{M}^{n+p}(c)$. Choose a local field of orthonormal frames

$$(2.1) \quad \begin{array}{c} e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}, \\ e_{1*} = J e_1, \dots, e_{n*} = J e_n, e_{(n+1)*} = J e_{n+1}, \dots, e_{(n+p)*} = J e_{n+p} \end{array}$$

in $\tilde{M}^{n+p}(c)$ in such a way that, restricted to M^n , e_1, \dots, e_n are tangent to M^n . For convenience, we use the following convention on the range

of indices:

$$\begin{aligned} A, B, C, \dots &= 1, \dots, n+p, 1^*, \dots, (n+p)^*; \\ i, j, k, \dots &= 1, \dots, n; \\ \alpha, \beta, \gamma, \dots &= n+1, \dots, n+p, 1^*, \dots, (n+p)^*; \\ \lambda, \mu, \dots &= n+1, \dots, n+p. \end{aligned}$$

With respect to the frame field of $\tilde{M}^{n+p}(c)$ chosen above, J has the component[8]

$$(2.2) \quad (J_{AB}) = \underbrace{\left(\begin{array}{c|c} \mathbf{0} & -I_{n+p} \\ \hline I_{n+p} & \mathbf{0} \end{array} \right)}_{\substack{i \quad \lambda \quad i^* \quad \lambda^*}} \left. \begin{array}{l} \}^i \\ \}^\lambda \\ \}^{i^*} \\ \}^{\lambda^*} \end{array} \right\}$$

where I_{n+p} denotes the identity matrix of degree $n+p$. Let $\{\omega^A\}$ be the dual frames of $\{e_A\}$, then the structure equations of $\tilde{M}^{n+p}(c)$ are given by

$$(2.3) \quad d\omega^A = \sum_B \omega^B \wedge \omega_B^A,$$

$$(2.4) \quad d\omega_B^A = \sum_C \omega_B^C \wedge \omega_C^A + \frac{1}{2} \sum_{C,D} K_{ABCD} \omega^C \wedge \omega^D,$$

where [8]

$$(2.5) \quad \begin{aligned} \omega_i^j &= \omega_{i^*}^{j^*}, \quad \omega_i^\lambda = \omega_{i^*}^{\lambda^*}, \quad \omega_i^{j^*} = \omega_j^{i^*}, \\ \omega_i^{\lambda^*} &= \omega_\lambda^{i^*}, \quad \omega_\lambda^\mu = \omega_{\lambda^*}^{\mu^*}, \quad \omega_\lambda^{\mu^*} = \omega_\mu^{\lambda^*}; \end{aligned}$$

$$(2.6) \quad K_{ABCD} = \frac{c}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}).$$

Restricting these forms to M^n , we have [8]

$$(2.7) \quad \omega^\alpha = 0,$$

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j, \quad h = \sum_{\alpha, i, j} h_{ij}^\alpha \omega^i \otimes \omega^j \otimes e_\alpha,$$

$$(2.8) \quad h_{jk}^{i^*} = h_{ik}^{j^*} = h_{ij}^{k^*},$$

$$(2.9) \quad \begin{cases} d\omega^i = \sum_j \omega^j \wedge \omega_j^i, \\ d\omega_j^i = \sum_k \omega_j^k \wedge \omega_k^i + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega^k \wedge \omega^l, \end{cases}$$

$$(2.10) \quad R_{ijkl} = K_{ijkl} + \sum_{\alpha} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha}),$$

$$(2.11) \quad d\omega_{\beta}^{\alpha} = \sum_{\gamma} \omega_{\beta}^{\gamma} \wedge \omega_{\gamma}^{\alpha} + \frac{1}{2} \sum_{k,l} R_{\alpha\beta kl} \omega^k \wedge \omega^l,$$

$$(2.12) \quad R_{\alpha\beta ij} = K_{\alpha\beta ij} + \sum_k (h_{ik}^{\alpha} h_{kj}^{\beta} - h_{jk}^{\alpha} h_{ki}^{\beta}),$$

where h is the second fundamental form of M^n , and R_{ijkl} , $R_{\alpha\beta ij}$ are the components of the Riemannian curvature tensor R and the normal curvature tensor R^{\perp} , respectively. We call that the normal connection is flat if $R^{\perp} = 0$. Let ζ be the mean curvature vector of M^n , i.e.,

$$\zeta = \frac{1}{n} \sum_{\alpha,j} h_{jj}^{\alpha} e_{\alpha}.$$

We call $|\zeta|$ the mean curvature of M^n , and denote it by H . From the equation of Gauss (2.10), we have [8]

$$(2.13) \quad \rho = n(n-1) \frac{c}{4} + n^2 H^2 - S,$$

where ρ is the scalar curvature of M^n , and $S = \sum_{\alpha,i,j} (h_{ij}^{\alpha})^2$. Define the first and the second covariant derivatives of h_{ij}^{α} as following

$$(2.14) \quad \sum_k h_{ijk}^{\alpha} \omega^k = dh_{ij}^{\alpha} + \sum_{\beta} h_{ij}^{\beta} \omega_{\beta}^{\alpha} - \sum_l h_{lj}^{\alpha} \omega_i^l - \sum_l h_{il}^{\alpha} \omega_j^l,$$

$$(2.15) \quad \sum_l h_{ijkl}^{\alpha} \omega^l = dh_{ijk}^{\alpha} + \sum_{\beta} h_{ijk}^{\beta} \omega_{\beta}^{\alpha} - \sum_l h_{ljk}^{\alpha} \omega_i^l - \sum_l h_{ilk}^{\alpha} \omega_j^l - \sum_l h_{ijl}^{\alpha} \omega_k^l,$$

then [8]

$$(2.16) \quad h_{ijk}^{\alpha} = h_{ikj}^{\alpha},$$

$$(2.17) \quad h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_m (h_{mi}^{\alpha} R_{mjkl} + h_{mj}^{\alpha} R_{mikl}) - \sum_{\beta} h_{ij}^{\beta} R_{\alpha\beta kl}.$$

From (2.16), (2.17), the Laplacian of h_{ij}^{α} is

$$(2.18) \quad \Delta h_{ij}^{\alpha} = \sum_k h_{k k i j}^{\alpha} + \sum_{k,m} (h_{im}^{\alpha} R_{mkjk} + h_{km}^{\alpha} R_{mijk}) - \sum_{\beta,k} h_{ki}^{\beta} R_{\alpha\beta jk}.$$

From (2.6), we have

Lemma 2.1. *Let M^n be a totally real submanifold of $\tilde{M}^{n+p}(c)$, then*

- (1) $K_{i^*j^*kl} = K_{ijkl} = \frac{c}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$;
- (2) $K_{\lambda Aij} = 0$, $K_{\lambda^* Aij} = 0$;
- (3) $K_{\alpha ijk} = 0$, $K_{\alpha \lambda jk} = 0$.

3. Position of umbilical normal vectors in the normal bundle

Now we assume that M^n is a totally real submanifold with flat normal connection in $\tilde{M}^{n+p}(c)$ ($c \neq 0$). For a point $x \in M^n$, $J(T_x M^n)$ is a subspace of the normal space $T_x^\perp M^n$, we denote it by V_x . Then $T_x^\perp M^n$ can be decomposed into the orthonormal sum $T_x^\perp M^n = V_x \oplus V_x^\perp$, where V_x^\perp is the orthonormal complement of V_x in $T_x^\perp M^n$. We can characterize the umbilical normal vectors by their position in the normal bundle as the following proposition.

Proposition 3.1. *Let M^n be a totally real submanifold with flat normal connection in $\tilde{M}^{n+p}(c)$ ($c \neq 0$), $x \in M^n$. Then a nonzero normal vector ξ is umbilical if and only if $\xi \in V_x^\perp$.*

Proof. Assume that ξ is a nonzero normal vector in V_x^\perp , we prove that the shape operator $A_{\frac{\xi}{|\xi|}}$ has exactly one eigenvalue. Choose the orthonormal frame (2.1) at the point x , such that e_1, \dots, e_n are eigenvectors of $A_{\frac{\xi}{|\xi|}}$ and $e_{n+1} = \frac{\xi}{|\xi|}$. Without loss of generality, we may assume that the eigenvalues of $A_{\frac{\xi}{|\xi|}}$ are given by

$$\lambda_1 = \dots = \lambda_{i_1} = \rho_1, \quad \lambda_{i_1+1} = \dots = \lambda_{i_1+i_2} = \rho_2, \dots, \\ \lambda_{i_1+\dots+i_{k-1}+1} = \dots = \lambda_n = \rho_k.$$

If ρ_1, \dots, ρ_k ($k \geq 2$) are all distinct, we will get a contradiction. Put

$$[\rho_1] = \{1, \dots, i_1\}, \quad [\rho_2] = \{i_1 + 1, \dots, i_1 + i_2\}, \dots, \\ [\rho_k] = \{i_1 + \dots + i_{k-1} + 1, \dots, n\}.$$

With respect to the frame chosen above, the matrix (h_{ij}^{n+1}) are given by

$$h_{ij}^{n+1} = \rho_s \delta_{ij}, \quad i, j \in [\rho_s], \quad s = 1, \dots, k, \\ h_{ij}^{n+1} = 0, \quad i \in [\rho_s], \quad j \in [\rho_t], \quad s \neq t.$$

Since the normal connection is flat, by using the equation of Ricci (2.12) and Lemma 2.1, we have

$$0 = R_{n+1\alpha ij} = K_{n+1\alpha ij} + \sum_l (h_{il}^{n+1} h_{lj}^\alpha - h_{il}^\alpha h_{lj}^{n+1}) \\ = \sum_l (h_{il}^{n+1} h_{lj}^\alpha - h_{il}^\alpha h_{lj}^{n+1}).$$

By setting $i \in [\rho_s]$, $j \in [\rho_t]$, $s \neq t$, the above equation becomes

$$(\rho_s - \rho_t)h_{ij}^\alpha = 0,$$

which implies that for all α ,

$$(3.1) \quad h_{ij}^\alpha = 0, \quad i \in [\rho_s], \quad j \in [\rho_t], \quad s \neq t.$$

By using (2.12) and Lemma 2.1 again, we also have

$$\begin{aligned} 0 &= R_{i^*j^*ij} = K_{i^*j^*ij} + \sum_l (h_{il}^{i^*} h_{lj}^{j^*} - h_{il}^{j^*} h_{lj}^{i^*}) \\ &= \frac{c}{4} + \sum_l (h_{il}^{i^*} h_{lj}^{j^*} - h_{il}^{j^*} h_{lj}^{i^*}). \end{aligned}$$

By setting $i \in [\rho_s]$, $j \in [\rho_t]$, $s \neq t$ in the above equation and considering (3.1), we get $c = 0$, which contradicts the fact $\tilde{M}^{n+p}(c)$ is non-flat.

Conversely, let ξ be an umbilical nonzero normal vector, it can be decomposed into the sum $\xi = \xi_1 + \xi_2$, where $\xi_1 \in V_x$, $\xi_2 \in V_x^\perp$. Choose the orthonormal frame (2.1) at the point x , such that

$$\xi_1 = |\xi_1|e_{1^*}, \quad \xi_2 = |\xi_2|e_{n+1}.$$

Noting that ξ is umbilical, we have

$$(3.2) \quad |\xi_1|h_{ij}^{1^*} + |\xi_2|h_{ij}^{n+1} = \langle \xi, \zeta \rangle \delta_{ij},$$

where \langle, \rangle denotes the inner product in $T_x^\perp M^n$. By using the equation of Ricci (2.12), Lemma 2.1 and considering the fact that the normal connection is flat, we obtain

$$(3.3) \quad \sum_l (h_{il}^{n+1} h_{lj}^{2^*} - h_{il}^{2^*} h_{lj}^{n+1}) = 0,$$

$$(3.4) \quad \sum_l (h_{il}^{1^*} h_{lj}^{2^*} - h_{il}^{2^*} h_{lj}^{1^*}) = \frac{c}{4}(\delta_{1j}\delta_{2i} - \delta_{1i}\delta_{2j}),$$

which, together with (3.2), imply that

$$\begin{aligned} 0 &= \sum_l (|\xi_1|h_{il}^{1^*} + |\xi_2|h_{il}^{n+1})h_{lj}^{2^*} - \sum_l h_{il}^{2^*}(|\xi_1|h_{lj}^{1^*} + |\xi_2|h_{lj}^{n+1}) \\ &= |\xi_1| \sum_l (h_{il}^{1^*} h_{lj}^{2^*} - h_{il}^{2^*} h_{lj}^{1^*}) + |\xi_2| \sum_l (h_{il}^{n+1} h_{lj}^{2^*} - h_{il}^{2^*} h_{lj}^{n+1}) \\ &= \frac{c}{4}|\xi_1|(\delta_{1j}\delta_{2i} - \delta_{1i}\delta_{2j}). \end{aligned}$$

By setting $i = 2, j = 1$ in the above equation and noting that $c \neq 0$, we get $|\xi_1| = 0$, hence $\xi = \xi_2 \in V_x^\perp$. \square

The above proposition can be verified by a classical example which appeared in [10, 17] as follows.

Example 3.1. Let $\mathbb{C}P^{n+p}$ be a complex projective space of constant holomorphic sectional curvature 4 and of complex dimension $n+p$. Let $S^m(r)$ be the Euclidean sphere of dimension m and radius r . Denote $S^m(1)$ just by S^m . It is well known that [18] $S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}})$ can be naturally immersed in S^5 as a minimal submanifold. Then, through the Hopf fibration $\pi : S^5 \rightarrow \mathbb{C}P^2$, one can get a minimal totally real submanifold $T^2 := \pi(S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}}))$ in $\mathbb{C}P^2$ in $\mathbb{C}P^{2+p}$ (see [10]). With respect to some suitable moving frames $\{e_A\}$ of the form (2.1), [10] and [17] calculated that

$$(3.5) \quad (h_{ij}^{1*}) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad (h_{ij}^{2*}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

$$(3.6) \quad h_{ij}^\alpha = 0, \quad \alpha \neq 1^*, 2^*, \quad i, j = 1, 2.$$

By using (3.5), (3.6), the equation of Ricci (2.12) and Lemma 2.1, one can easily find that the normal connection of T^2 in $\mathbb{C}P^{2+p}$ is flat. Also (3.5) and (3.6) implies that for any point $x \in T^2$, the normal vectors in V_x are not umbilical, while the normal vectors in V_x^\perp are all umbilical (in fact, in this minimal case, they are all geodesic).

From Proposition 3.2 one can easily get the following corollary.

Corollary 3.2. *In non-flat complex space forms, there exist no totally umbilical totally real submanifolds with flat normal connections.*

Combining Corollary 3.2 and a classification result of totally umbilical submanifolds in non-flat complex space forms (see Theorem 1 of [19]), we have the following

Corollary 3.3. *Let M^n be a totally umbilical submanifold in a non-flat complex space form $\tilde{M}^{n+p}(c)$. If the normal connection of M^n is flat, then M^n must be a complex space form immersed holomorphically in $\tilde{M}^{n+p}(c)$ as a totally geodesic submanifold.*

4. Proof of the theorem

Now we assume that M^n is a pseudo-umbilical totally real submanifold with flat normal connection in $\tilde{M}^{n+p}(c)$ ($c \neq 0$). If M^n is not minimal, then there exists a point $x_0 \in M^n$, such that $H(x_0) \neq 0$. From Proposition 3.1 we may choose moving frame (2.1) such that $\zeta = He_{n+1}$ around the point x_0 , so

$$(4.1) \quad \sum_j h_{jj}^{n+1} = nH, \quad \sum_j h_{jj}^\alpha = 0 \quad (\alpha \neq n+1),$$

from which, combined with Proposition 3.1, we know that

$$(4.2) \quad h_{ij}^{n+1} = H\delta_{ij}, \quad h_{ij}^\alpha = 0 \quad (\alpha \neq 1^*, \dots, n^*, n+1),$$

which are equivalent to

$$(4.3) \quad \omega_i^{n+1} = H\omega^i, \quad \omega_i^\alpha = 0, \quad (\alpha \neq 1^*, \dots, n^*, n+1).$$

For convenience, we denote that $\sigma = S - nH^2$. The proof of the theorem is based on the calculation of the Laplacian of σ . From (2.18), and noting that the normal connection is flat, we have

$$(4.4) \quad \begin{aligned} \frac{1}{2}\Delta\sigma &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,k,\alpha} h_{ij}^\alpha h_{kkij}^\alpha - \frac{n}{2}\Delta H^2 \\ &+ \sum_{i,j,k,m,\alpha} h_{ij}^\alpha (h_{im}^\alpha R_{mkjk} + h_{km}^\alpha R_{mijk}). \end{aligned}$$

Write

$$\begin{aligned} \text{I} &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2, \\ \text{II} &= \sum_{i,j,k,\alpha} h_{ij}^\alpha h_{kkij}^\alpha - \frac{n}{2}\Delta H^2, \\ \text{III} &= \sum_{i,j,k,m,\alpha} h_{ij}^\alpha (h_{im}^\alpha R_{mkjk} + h_{km}^\alpha R_{mijk}). \end{aligned}$$

We will calculate I, II and III in turn.

From (2.14) and (4.1) one can easily get

$$(4.5) \quad \sum_i h_{iik}^{n+1} = nH_k, \quad \sum_i h_{iik}^\alpha = nH\omega_{n+1}^\alpha(e_k) \quad (\alpha \neq n+1),$$

where $H_k = e_k(H)$. Furthermore, by using (2.5) and (4.2), we deduce that

$$(4.6) \quad \sum_i h_{iik}^{m*} = nH\omega_{n+1}^{m*}(e_k) = nH\omega_m^{(n+1)*}(e_k) = nHh_{mk}^{(n+1)*} = 0.$$

Lemma 4.1. *Let $\text{grad } H$ denote the gradient of H , then*

$$\text{I} = \sum_{i,j,k,m} (h_{ijk}^{m*})^2 + H^2\sigma + \frac{1}{n} \sum_{\alpha \neq n+1} \sum_k \left(\sum_i h_{iik}^\alpha \right)^2 + n|\text{grad } H|^2.$$

Proof. From (2.14) and (4.2), we have

$$h_{ijk}^\alpha = e_k(h_{ij}^\alpha) + \sum_m h_{ij}^{m*} \omega_m^\alpha(e_k) + H\delta_{ij}\omega_{n+1}^\alpha(e_k) - \sum_l h_{lj}^\alpha \omega_i^l(e_k) - \sum_l h_{il}^\alpha \omega_j^l(e_k).$$

Applying (2.5) and (4.2) to the above formula, it follows that

$$h_{ijk}^{n+1} = H_k \delta_{ij} + \sum_m h_{ij}^{m*} \omega_m^{n+1}(e_k) = H_k \delta_{ij} - \sum_m h_{ij}^{m*} \omega_m^{(n+1)*}(e_k) = H_k \delta_{ij},$$

$$h_{ijk}^{(n+1)*} = H h_{ij}^{k*} + H \delta_{ij} \omega_{n+1}^{(n+1)*}(e_k),$$

$$h_{ijk}^\alpha = H \delta_{ij} \omega_{n+1}^\alpha(e_k), \quad (\alpha \neq n+1, 1^*, \dots, (n+1)^*).$$

From which, together with (4.2) and (4.5), we may calculate I as follows.

$$\begin{aligned} \text{I} &= \sum_{i,j,k,m} (h_{ijk}^{m*})^2 + n H^2 \sum_{\alpha \neq n+1, 1^*, \dots, (n+1)^*} \sum_k (\omega_{n+1}^\alpha(e_k))^2 + n |\text{grad } H|^2 \\ &\quad + H^2 \sum_{i,j,k} \left(h_{ij}^{k*} + \delta_{ij} \omega_{n+1}^{(n+1)*}(e_k) \right)^2 \\ &= \sum_{i,j,k,m} (h_{ijk}^{m*})^2 + n |\text{grad } H|^2 + n H^2 \sum_{\alpha \neq n+1, 1^*, \dots, (n+1)^*} \sum_k (\omega_{n+1}^\alpha(e_k))^2 + H^2 \sigma \\ &\quad + 2 H^2 \sum_{i,k} h_{ii}^{k*} \omega_{n+1}^{(n+1)*}(e_k) + n H^2 \sum_k (\omega_{n+1}^{(n+1)*}(e_k))^2 \\ &= \sum_{i,j,k,m} (h_{ijk}^{m*})^2 + n |\text{grad } H|^2 + H^2 \sigma + n H^2 \sum_{\alpha \neq n+1, 1^*, \dots, n^*} \sum_k (\omega_{n+1}^\alpha(e_k))^2 \\ &= \sum_{i,j,k,m} (h_{ijk}^{m*})^2 + n |\text{grad } H|^2 + H^2 \sigma + \frac{1}{n} \sum_{\alpha \neq n+1, 1^*, \dots, n^*} \sum_k \left(\sum_i h_{iik}^\alpha \right)^2. \end{aligned}$$

This completes the proof. \square

Lemma 4.2. $\text{II} = -\frac{1}{n} \sum_{\alpha \neq n+1} \sum_k \left(\sum_i h_{iik}^\alpha \right)^2 - n |\text{grad } H|^2.$

Proof. From (4.2),

$$\sum_{\alpha, i, j, k} h_{ij}^\alpha h_{kkij}^\alpha = \sum_{m, i, j, k} h_{ij}^{m*} h_{kkij}^{m*} + \sum_{i, j, k} h_{ij}^{n+1} h_{kkij}^{n+1}.$$

We compute each term of the right hand side of the above formula as follows. During the procedure, we have used the formulas (2.5), (2.8),

(4.2) and (4.6).

$$\begin{aligned}
\sum_{m,i,j,k} h_{ij}^{m*} h_{kki}^{m*} &= \sum_{m,i,j,k} h_{ij}^{m*} \left(e_j(h_{kki}^{m*}) + \sum_{\beta} h_{kki}^{\beta} \omega_{\beta}^{m*}(e_j) - \sum_l h_{lki}^{m*} \omega_k^l(e_j) \right. \\
&\quad \left. - \sum_l h_{kli}^{m*} \omega_k^l(e_j) - \sum_l h_{kkl}^{m*} \omega_i^l(e_j) \right) \\
&= \sum_{m,i,j,k} \sum_{\beta} h_{ij}^{m*} h_{kki}^{\beta} \omega_{\beta}^{m*}(e_j) \\
&= \sum_{m,i,j,k,l} h_{ij}^{m*} h_{kki}^{l*} \omega_l^{m*}(e_j) + \sum_{m,i,j,k} h_{ij}^{m*} h_{kki}^{n+1} \omega_{n+1}^{m*}(e_j) \\
&\quad + \sum_{m,i,j,k} \sum_{\lambda \neq n+1} h_{ij}^{m*} h_{kki}^{\lambda} \omega_{\lambda}^{m*}(e_j) + \sum_{m,i,j,k} \sum_{\lambda} h_{ij}^{m*} h_{kki}^{\lambda*} \omega_{\lambda}^{m*}(e_j) \\
&= - \sum_{m,i,j,k} \sum_{\lambda} h_{ij}^{m*} h_{kki}^{\lambda*} \omega_{\lambda}^{m*}(e_j) \\
&= -H \sum_{m,i,j,k} h_{ij}^{m*} \delta_{mj} h_{kki}^{(n+1)*} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j,k} h_{ij}^{n+1} h_{kki}^{n+1} &= \sum_{i,j,k} H \delta_{ij} \left(e_j(h_{kki}^{n+1}) + \sum_{\beta} h_{kki}^{\beta} \omega_{\beta}^{n+1}(e_j) - \sum_l h_{lki}^{n+1} \omega_k^l(e_j) \right. \\
&\quad \left. - \sum_l h_{kli}^{n+1} \omega_k^l(e_j) - \sum_l h_{kkl}^{n+1} \omega_i^l(e_j) \right) \\
&= \sum_{i,j} H \delta_{ij} \left(e_j(nH_i) - nH_l \omega_i^l(e_j) - \frac{1}{nH} \sum_{\beta \neq n+1} \left(\sum_k h_{kki}^{\beta} \right) \left(\sum_l h_{llj}^{\beta} \right) \right) \\
&= \sum_{i,j} H \delta_{ij} \left(nH_{ij} - \frac{1}{nH} \sum_{\beta \neq n+1} \left(\sum_k h_{kki}^{\beta} \right) \left(\sum_l h_{llj}^{\beta} \right) \right) \\
&= nH \Delta H - \frac{1}{n} \sum_{\beta \neq n+1} \sum_i \left(\sum_k h_{kki}^{\beta} \right)^2.
\end{aligned}$$

Noting that

$$\Delta H^2 = 2H \Delta H + 2|\text{grad } H|^2,$$

hence

$$\Pi = -\frac{1}{n} \sum_{\alpha \neq n+1} \sum_i \left(\sum_k h_{kki}^{\alpha} \right)^2 - n|\text{grad } H|^2.$$

□

Lemma 4.3. $\text{III} = nH^2\sigma$.

Proof. Applying Lemma 2.1, (4.2), (2.8) and (2.12) to the equation of Gauss (2.10), noting that the normal connection is flat, we have

$$\begin{aligned}
 R_{ijkl} &= K_{ijkl} + \sum_{\alpha} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha}) \\
 &= K_{i^*j^*kl} + \sum_m (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}) + H^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
 &= K_{i^*j^*kl} + \sum_m (h_{km}^{i^*} h_{ml}^{j^*} - h_{lm}^{i^*} h_{mk}^{j^*}) + H^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
 &= R_{i^*j^*kl} + H^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
 (4.7) \quad &= H^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).
 \end{aligned}$$

Then Lemma 4.3 follows by a direct calculation. □

Proof of Theorem 1.1. Assume M^n is not minimal, then there exists a point $x_0 \in M^n$, such that $H(x_0) \neq 0$. Applying Lemma 4.1, Lemma 4.2 and Lemma 4.3 to (4.4), we have

$$(4.8) \quad \frac{1}{2}\Delta\sigma = \sum_{m,i,j,k} (h_{ijk}^{m^*})^2 + (n+1)H^2\sigma.$$

On the other hand, (4.7) implies that the scalar curvature of M^n is given by

$$\rho = n(n-1)H^2,$$

substituted into (2.13), we see that $\sigma = \frac{c}{4}n(n-1)$. Thus (4.8) becomes

$$0 = \sum_{m,i,j,k} (h_{ijk}^{m^*})^2 + \frac{c}{4}n(n^2-1)H^2.$$

Hence $H = 0$ around the point x_0 , which contradicts the assumption $H(x_0) \neq 0$. Therefore, M^n must be minimal. This completes the proof. □

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REFERENCES

- [1] B. Y. Chen, Classification of Lagrangian surfaces of constant curvature in complex projective plane, *J. Geom. Phys.*, **53** (2005), 428-460.
- [2] B. Y. Chen, Three additional families of Lagrangian surfaces of constant curvature in complex projective plane, *J. Geom. Phys.*, **56** (2006), 666-669.
- [3] B. Y. Chen, Classification of Lagrangian surfaces of constant curvature in complex hyperbolic plane, *J. Geom. Phys.*, **55** (2005), 399-439.
- [4] B. Y. Chen, Classification of Lagrangian surfaces of constant curvature in complex hyperbolic plane. Part II, *Soochow J. Math.*, **33** (2007), 127-165.
- [5] B. Y. Chen, Classification of Lagrangian surfaces of constant curvature in the complex Euclidean plane, *Proc. Edinburgh Math. Soc. (Series 2)*, **48** (2005), 337-364.
- [6] B. Y. Chen, F. Dillen, Classification of marginally trapped Lagrangian surfaces in Lorentzian complex space forms, *J. Math. Phys.*, **48** (2007), no.1, 013509.
- [7] S. C. Shu, A.Y. Han, Extremal Lagrangian submanifolds in a complex space form $N^n(4c)$, *Turk. J. Math.*, **34** (2010), 129-143.
- [8] B. Y. Chen and K. Ogiue, On totally real submanifolds, *Trans. Amer. Math. Soc.*, **193** (1974), 257-266.
- [9] K. Yano and M. Kon, Totally real submanifolds of complex space forms, I, *Tohoku Math. J., Second Series*, **28** (1976), 215-225.
- [10] K. Yano and M. Kon, Totally real submanifolds of complex space forms, II, *Kodai Mathematical Seminar Reports*, **27** (1976), 385-399.
- [11] B. Y. Chen, C. S. Houh and H. S. Lue, Totally real submanifolds, *J. Differ. Geom.*, **12** (1977), 473-480.
- [12] F. Urbano, Nonnegatively curved totally real submanifolds, *Math. Ann.*, **273** (1986), 345-348.
- [13] F. Urbano, Totally real submanifolds, *Geometry and Topology of Submanifolds*, World Scientific, Singapore, 1989, 198-208.
- [14] B. Y. Chen and K. Ogiue, Classification of slumbilical submanifolds in complex space forms, *Osaka. J. Math.*, **39** (2002), 23-47.
- [15] L. Zhang, Some remarks on pseudo-umbilical totally real submanifolds in a complex projective space, *Appl. Math. J. Chinese Univ.*, **23** (2008) 227-232.
- [16] L. Zhang and W. D. Song, Totally real pseudo-umbilical submanifolds with flat normal bundle of complex projective space, *Acta Math. Sci. Ser. A*, **27** (2007), 688-695 (in Chinese).
- [17] G. D. Ludden, M. Okumura and K. Yano, A totally real surface in $\mathbb{C}P^2$ that is not totally geodesic, *Proc. Amer. Math. Soc.*, **53** (1975), 186-190.
- [18] S. S. Chern, M. do Carmo and S. Kobayashi, Minimal Submanifolds of a Sphere with Second Fundamental Form of Constant Length, *Functional Analysis and Related Fields*. Springer Berlin Heidelberg, 1970, 59-75.
- [19] B. Y. Chen and K. Ogiue, Two theorems on Kaehler manifolds, *Michigan Math. J.*, **21** (1974), 225-229.

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